Given that

$$\mathrm{p}(\mathsf{t}|\boldsymbol{w}) = \prod_{n=1}^{N} \mathcal{N}(t_n|\boldsymbol{w}^\top \boldsymbol{\phi}(\boldsymbol{x}_n), \beta^{-1}) \tag{3.10}$$

$$\mathrm{p}(\boldsymbol{w}) = \mathcal{N}(\boldsymbol{w}|\boldsymbol{m}_0, \boldsymbol{S}_0) \tag{3.48}$$

Now we want to calculate the posterior distribution through

$$\begin{aligned}
\mathrm{p}(\boldsymbol{w}|\mathsf{t}) =& \frac{\mathrm{p}(\mathsf{t}|\boldsymbol{w})\mathrm{p}(\boldsymbol{w})}{\int \mathrm{p}(\mathsf{t}|\boldsymbol{w})\mathrm{p}(\boldsymbol{w})\mathrm{d}\boldsymbol{w}} \\
\Leftrightarrow \ln \mathrm{p}(\boldsymbol{w}|\mathsf{t}) =& \ln \mathrm{p}(\mathsf{t}|\boldsymbol{w}) + \ln \mathrm{p}(\boldsymbol{w}) \underbrace{- \ln \left(\int \mathrm{p}(\mathsf{t}|\boldsymbol{w})\mathrm{p}(\boldsymbol{w})\mathrm{d}\boldsymbol{w} \right)}_{\text{const.}} \\
=& \ln \mathrm{p}(\mathsf{t}|\boldsymbol{w}) + \ln \mathrm{p}(\boldsymbol{w}) + \text{const.} \\
=& \underbrace{-\frac{N}{2} \ln 2\pi - \frac{N}{2} \ln \beta}_{\text{const.}} - \sum_{n=1}^{N} \frac{\beta}{2} \{t_n - \boldsymbol{w}^\top \boldsymbol{\phi}(\boldsymbol{x}_n)\}^2 \\
& \underbrace{-\frac{D}{2} \ln 2\pi - \frac{1}{2} \ln |\boldsymbol{S}_0|}_{\text{const.}} - \frac{1}{2} (\boldsymbol{w} - \boldsymbol{m}_0)^\top \boldsymbol{S}_0^{-1} (\boldsymbol{w} - \boldsymbol{m}_0) + \text{const.} \\
=& -\frac{1}{2} \sum_{n=1}^{N} \underbrace{\beta t_n^2}_{\text{const.}} - 2\beta t_n \boldsymbol{w}^\top \boldsymbol{\phi}(\boldsymbol{x}_n) + \beta \boldsymbol{w}^\top \boldsymbol{\phi}(\boldsymbol{x}_n) \boldsymbol{\phi}(\boldsymbol{x}_n)^\top \boldsymbol{w} \\
& - \frac{1}{2} \boldsymbol{w}^\top \boldsymbol{S}_0^{-1} \boldsymbol{w} + \frac{1}{2} 2 \boldsymbol{m}_0^\top \boldsymbol{S}_0^{-1} \boldsymbol{w} - \underbrace{\frac{1}{2} \boldsymbol{m}_0^\top \boldsymbol{S}_0^{-1} \boldsymbol{m}_0}_{\text{const.}} + \text{const.} \\
=& -\frac{1}{2} \sum_{n=1}^{N} \boldsymbol{w}^\top \left[-2\beta t_n \boldsymbol{\phi}(\boldsymbol{x}_n) \right] - \frac{1}{2} \boldsymbol{w}^\top \left[\sum_{n=1}^{N} \beta \boldsymbol{\phi}(\boldsymbol{x}_n) \boldsymbol{\phi}(\boldsymbol{x}_n)^\top \right] \boldsymbol{w} \\
& - \frac{1}{2} \boldsymbol{w}^\top \boldsymbol{S}_0^{-1} \boldsymbol{w} + \frac{1}{2} 2 \boldsymbol{w}^\top \boldsymbol{S}_0^{-1} \boldsymbol{m}_0 + \text{const.} \\
=& -\frac{1}{2} \boldsymbol{w}^\top \left[-2\beta \boldsymbol{\Phi}^\top \mathsf{t} - 2 \boldsymbol{S}_0^{-1} \boldsymbol{m}_0 \right] \\
& - \frac{1}{2} \boldsymbol{w}^\top \left[\beta \boldsymbol{\Phi}^\top \boldsymbol{\Phi} - \boldsymbol{S}_0^{-1} \right] \boldsymbol{w} + \text{const.}
\end{aligned}$$

By comparing with

$$\begin{aligned}
\ln \mathrm{p}(\boldsymbol{w}|\mathsf{t}) =& -\frac{1}{2} (\boldsymbol{w} - \boldsymbol{m}_N)^\top \boldsymbol{S}_N^{-1} (\boldsymbol{w} - \boldsymbol{m}_N) + \text{const.} \\
=& -\frac{1}{2} \boldsymbol{w}^\top \boldsymbol{S}_N^{-1} \boldsymbol{w} - \frac{1}{2} \boldsymbol{w}^\top \left[-2 \boldsymbol{S}_N^{-1} \boldsymbol{m}_N \right] + \text{const.}
\end{aligned}$$

We know directly

$$\boldsymbol{S}_N^{-1} = \beta \boldsymbol{\Phi}^\top \boldsymbol{\Phi} - \boldsymbol{S}_0^{-1}$$

and

$$\begin{aligned}
2 \boldsymbol{S}_N^{-1} \boldsymbol{m}_N &= 2\beta \boldsymbol{\Phi}^\top \mathsf{t} + 2 \boldsymbol{S}_0^{-1} \boldsymbol{m}_0 \\
\Leftrightarrow \qquad \boldsymbol{m}_N &= \boldsymbol{S}_N \left(\beta \boldsymbol{\Phi}^\top \mathsf{t} + \boldsymbol{S}_0^{-1} \boldsymbol{m}_0 \right)
\end{aligned}$$

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