We know from Exercise 3.8-9 that

$$\begin{aligned}\boldsymbol{S}_{N+1} &= [\boldsymbol{S}_N^{-1} + \beta^{\frac{1}{2}}\boldsymbol{\phi}(\boldsymbol{x}_{N+1})\beta^{\frac{1}{2}}\boldsymbol{\phi}(\boldsymbol{x}_{N+1})^\top]^{-1} \\ &= \boldsymbol{S}_N - \frac{\beta \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x}_{N+1})\boldsymbol{\phi}(\boldsymbol{x}_{N+1})^\top \boldsymbol{S}_N}{1 + \beta\boldsymbol{\phi}(\boldsymbol{x}_{N+1})^\top \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x}_{N+1})}\end{aligned}$$

So we have

$$\begin{aligned}\sigma_{N+1}^2(\boldsymbol{x}) &= \frac{1}{\beta} + \boldsymbol{\phi}(\boldsymbol{x})^\top \boldsymbol{S}_{N+1}\boldsymbol{\phi}(\boldsymbol{x}) \\ &= \frac{1}{\beta} + \boldsymbol{\phi}(\boldsymbol{x})^\top \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x}) - \frac{\beta\boldsymbol{\phi}(\boldsymbol{x})^\top \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x}_{N+1})\boldsymbol{\phi}(\boldsymbol{x}_{N+1})^\top \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x})}{1 + \beta\boldsymbol{\phi}(\boldsymbol{x}_{N+1})^\top \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x}_{N+1})} \\ &= \sigma_N^2(\boldsymbol{x}) - \frac{\beta[\boldsymbol{\phi}(\boldsymbol{x})^\top \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x}_{N+1})]^2}{1 + \beta\boldsymbol{\phi}(\boldsymbol{x}_{N+1})^\top \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x}_{N+1})}\end{aligned}$$

Considering that the covariance matrix $\boldsymbol{S}_N$ is positive semi-definite, which means that

$$\forall \boldsymbol{x} \in \mathbb{R}^M, \boldsymbol{x}^\top \boldsymbol{S}_N \boldsymbol{x} \geqslant 0$$

making

$$\frac{\beta[\boldsymbol{\phi}(\boldsymbol{x})^\top \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x}_{N+1})]^2}{1 + \beta\boldsymbol{\phi}(\boldsymbol{x}_{N+1})^\top \boldsymbol{S}_N \boldsymbol{\phi}(\boldsymbol{x}_{N+1})} \geqslant 0$$

and

$$\sigma_{N+1}^2(\boldsymbol{x}) \leqslant \sigma_N^2(\boldsymbol{x})$$

■