For an arbitrary square matrix $\mathbf{w}$, suppose that there exist a symmetric matrix $\mathbf{w}^{\mathrm{S}}$ and an anti-symmetric matrix $\mathbf{w}^{\mathrm{A}}$ s.t.

$$\mathbf{w} = \mathbf{w}^{\mathrm{S}} + \mathbf{w}^{\mathrm{A}} \tag{1}$$

we also have

$$\begin{aligned} \mathbf{w}^{\intercal} &= (\mathbf{w}^{\mathrm{S}})^{\intercal} + (\mathbf{w}^{\mathrm{A}})^{\intercal} \\ &= \mathbf{w}^{\mathrm{S}} - \mathbf{w}^{\mathrm{A}} \end{aligned} \tag{2}$$

By solving equations (1) and (2), we obtain a unique solution:

$$\mathbf{w}^{\mathrm{S}} = \frac{\mathbf{w} + \mathbf{w}^{\intercal}}{2}, \mathbf{w}^{\mathrm{A}} = \frac{\mathbf{w} - \mathbf{w}^{\intercal}}{2}$$

One can easily verify that

$$\begin{aligned} (\mathbf{w}^{\mathrm{S}})^{\intercal} &= \frac{\mathbf{w}^{\intercal} + (\mathbf{w}^{\intercal})^{\intercal}}{2} = \frac{\mathbf{w}^{\intercal} + \mathbf{w}}{2} = \mathbf{w}^{\mathrm{S}} \\ (\mathbf{w}^{\mathrm{A}})^{\intercal} &= \frac{\mathbf{w}^{\intercal} - (\mathbf{w}^{\intercal})^{\intercal}}{2} = -\frac{\mathbf{w} - \mathbf{w}^{\intercal}}{2} = -\mathbf{w}^{\mathrm{A}} \end{aligned}$$

which means that the assumption is valid.

Given that $w_{ij} = w_{ij}^{\mathrm{S}} + w_{ij}^{\mathrm{A}}$,

$$\begin{aligned} &\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}x_ix_j \\ =&\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^{\mathrm{S}}x_ix_j + \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^{\mathrm{A}}x_ix_j \\ =&\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^{\mathrm{S}}x_ix_j + \underbrace{\sum_{i=1}^{D} w_{ii}^{\mathrm{A}}x_ix_i}_{\text{trace is 0}} + \sum_{i=2}^{D}\sum_{j=1}^{i-1} w_{ij}^{\mathrm{A}}x_ix_j + \sum_{j=2}^{D}\sum_{i=1}^{j-1} w_{ij}^{\mathrm{A}}x_ix_j \\ =&\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^{\mathrm{S}}x_ix_j + \sum_{i=2}^{D}\sum_{j=1}^{i-1} w_{ij}^{\mathrm{A}}x_ix_j + \sum_{i=2}^{D}\sum_{j=1}^{i-1} w_{ji}^{\mathrm{A}}x_jx_i \\ =&\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^{\mathrm{S}}x_ix_j + \sum_{i=2}^{D}\sum_{j=1}^{i-1} w_{ij}^{\mathrm{A}}x_ix_j - \sum_{i=2}^{D}\sum_{j=1}^{i-1} w_{ij}^{\mathrm{A}}x_ix_j \\ =&\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^{\mathrm{S}}x_ix_j \end{aligned}$$

The number of independent parameters corresponds to the number of parameters of the triangular part of the matrix, that is

$$\sum_{i=1}^{D} i = \frac{D(D+1)}{2}$$

■